Lissajous Figures and Chebyshev Polynomials

Julio Castiñeira Merino

Julio Castiñeira Merino (jcastine@boj.pntic.mec.es) graduated with a degree in mathematics from the Universidad Complutense in Madrid, Spain. His academic interests include geometry, numerical analysis and computer science. He plays golf in his free time and enjoys music and movies, as well as spending time with family and friends.

To my wife Julia and my children Isabel, Lucia, and Rodrigo.

The aim of this paper is to obtain implicit equations for Lissajous figures in the form $f(x, y) = 0$ using some elementary properties of Chebyshev polynomials.

A Lissajous figure is the trajectory of a moving point whose rectangular coordinates are simple harmonic motions. The equation of a simple harmonic motion is $x = a\cos(\omega t + \varphi)$ where t is the time and the constants a, ω, and φ are the amplitude, the frequency, and the phase respectively. Parametric equations for Lissajous figures thus are

$$x = a\cos(\omega_1 t + \varphi_1)$$
$$y = b\cos(\omega_2 t + \varphi_2).$$

The constants a and b determine the size of the curve while its shape depends on the ratio of the frequencies. If the frequencies are equal, the curve is either an ellipse or a line segment, the latter occurring if the difference of the phases is a multiple of π. This property can be used to study an unknown signal. If we apply the unknown signal to the vertical axis of an oscilloscope and then vary the horizontal frequency, when the oscilloscope shows an ellipse the signal's frequency has been determined.

Lissajous figures were actually discovered by the American astronomer and mathematician Nathaniel Bowditch in 1815 when he was studying the movement of a compound pendulum. Bowditch (1773–1838) was a self-taught scientist, a captain of a merchant ship, president of an insurance company, actuary for the Massachusetts Hospital Insurance Company of Boston, and president of the American Academy of Arts and Sciences. Author of a number of scientific works, he is remembered mostly for his book *The New American Practical Navigator*, which was adopted by the U.S. Department of the Navy. Jules Antoine Lissajous (1822–1880) was a French physicist who extensively studied the curves that bear his name independently of Bowditch during his research on optics [**2**].

Among the different ways of writing the parametric equations for Lissajous figures, or curves (a more natural term), we choose the following, where m and n are integers, prime to each other, p and q are real numbers, and $0 \le t \le 2\pi$:

$$\begin{aligned} x &= a\cos(mt + p) \\ y &= b\sin(nt + q). \end{aligned} \tag{1}$$

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Obviously any figure whose ratio of frequencies is rational can be expressed in this way by making a linear change of variable. Expressing the abscissa as a function of the sine and the ordinate as a function of the cosine is done for a purpose. It is like the parametrization of the circle and it allows us to express the equation of a Lissajous figure in a simple way.

We have already stated that the ellipse is a Lissajous curve. Other well-known curves are Lissajous curves as well. We list some of them with their names [**3**]. Readers with graphing calculators can see their shapes, and those of other Lissajous curves not listed.

Parametric equation	Name	Implicit equation
$x = r\cos t$ $y = r\sin t$	Circle	$x^2 + y^2 = r^2$
$x = a\cos(t - \pi/2)$ $y = b\sin t$	Line segment	$x/a - y/b = 0$
$x = a\cos t$ $y = b\sin t$	Ellipse	$x^2/a^2 + y^2/b^2 = 1$
$x = a\cos t$ $y = (a/2)\sin 2t$	Lemniscate of Gerono	$x^4 = a^2(x^2 - y^2)$
$x = -\cos 2t$ $y = 2\sqrt{p}\sin t$	Letter C (Parabola)	$y^2 = 2p(x+1)$
$x = \cos 2t$ $y = \sin(3t + \pi/2)$	Tschirnhausen cubic	$2y^2 = 4x^3 - 3x + 1$
$x = \cos t$ $y = \sin 3t$	ABC's logo	$(4x^3 - 3x)^2 + y^2 = 1$
$x = 2\sqrt{2}\cos t$ $y = \sqrt{2}\sin(2t + 5\pi/4)$	Saddlebag	$x^4 + 4x^2(y-2) + 8(y-1)^2 = 0$

These examples are useful to point out some of the properties of the Lissajous curves. For instance, in the case of the segment, the parabola, or the Tchirhausen cubic the moving point turns around and runs the same course back. We will refer to those instances as degenerate cases. In the non-degenerate cases, the point never reverses direction. The terminology may not be the best but is useful to identify the cases readily. A Lissajous curve (1) is degenerate when either m is even and $\sin\delta = 0$ or m is odd and $\cos\delta = 0$ where $\delta = mq - np$. In the degenerate case, the curve can be determined with t varying in an interval of length π. But not every interval of this length can describe the curve completely. This discussion about the parameter is important if in the definition of parametric curve the condition of being locally injective is included [**1**], [**4**].

In analytic geometry a plane curve can be defined by its implicit equation $f(x, y) = 0$. If $f(x, y) = \sum a_{ij}x^i y^j$ is a polynomial, it is called an algebraic curve and a transcendental curve if f is a transcendental function. Lissajous curves are algebraic curves.

The implicit equation of a Lissajous curve is obtained by eliminating the parameter t from the parametric equations. This process can be done as follows:

1. Express x and y in terms of $\cos mt$, $\sin mt$, $\cos nt$, and $\sin nt$ using the sum and difference of angles formulas.
2. Express the trigonometric functions of multiple angles in terms of $\sin t$ and $\cos t$.
3. Express the functions $\sin t$ and $\cos t$ in terms of $u = \tan(t/2)$ using the familiar formulas
$$\sin t = \frac{2u}{1+u^2}, \cos t = \frac{1-u^2}{1+u^2}.$$
We then have a rational parametrization of the curve, $x = r(u)$, $y = s(u)$.
4. Eliminate the parameter u.

The above process can be done, sometimes simply, but most of the time in a complicated way that requires time and the use of tools of algebraic elimination [**5**]. This difficulty becomes apparent when one applies this process to some of the parametric equations listed above. We will prove that the implicit equations of Lissajous curves can be found using Chebyshev polynomials.

The Chebyshev polynomial of degree n is characterized by the property $T_n(\cos a) = \cos na$. The first six polynomials with their corresponding multiple-angle formulas are

Polynomial	Multiple-angle formula
$T_0(x) = 1$	$\cos 0 \cdot a = 1$
$T_1(x) = x$	$\cos a = \cos a$
$T_2(x) = 2x^2 - 1$	$\cos 2a = 2\cos^2 a - 1$
$T_3(x) = 4x^3 - 3x$	$\cos 3a = 4\cos^3 a - 3\cos a$
$T_4(x) = 8x^4 - 8x^2 + 1$	$\cos 4a = 8\cos^4 a - 8\cos^2 a + 1$
$T_5(x) = 16x^5 - 20x^3 + 5x$	$\cos 5a = 16\cos^5 a - 20\cos^3 a + 5\cos a$

When n is small, the Chebyshev polynomials can be computed by using the recurrence $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$, $T_0(x) = 1$, $T_1(x) = x$. The formula can be obtained by using the formula for the sum of cosines

$$T_n(\cos a) + T_{n-2}(\cos a) = \cos na + \cos(n-2)a = 2\cos a \cos(n-1)a.$$

Chebyshev polynomials are Lissajous curves. Indeed, a parametrisation of the curve $y = T_n(x)$ for $|x| \le 1$ is $x = \cos t$, $y = -\sin(nt - \pi/2)$, $0 \le t \le \pi$.

Chebyshev polynomials have the property

$$T_n(\sin a) = \begin{cases} (-1)^{(n-1)/2} \sin na, & n \text{ odd} \\ (-1)^{n/2} \cos na, & n \text{ even.} \end{cases} \tag{2}$$

To see this when $n = 2k$,

$$\begin{aligned} T_{2k}(\sin a) &= T_{2k}(\cos(a - \pi/2)) = \cos 2k(a - \pi/2) = \cos(2ka - k\pi) \\ &= \cos 2ka \cos k\pi + \sin 2ka \sin k\pi = (-1)^k \cos 2ka. \end{aligned}$$

The proof is similar when n is odd.

Theorem. Given the parametric plane curve

$$x = \cos(mt + p)$$
$$y = \sin(nt + q),$$

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$0 \le t \le 2\pi$, where p, q are real numbers and m, n integers prime to each other. Let

$$\delta = \begin{vmatrix} m & p \\ n & q \end{vmatrix}.$$

Then the equation of the curve is

$$T_n(x)^2 + T_m(y)^2 - 2(-1)^{m/2} T_n(x) T_m(y) \cos\delta - \sin^2\delta = 0 \tag{3}$$

if m is even and $\sin\delta \neq 0$,

$$\cos\delta T_n(x) - (-1)^{m/2} T_m(y) = 0 \tag{4}$$

if m is even and $\sin\delta = 0$,

$$T_n(x)^2 + T_m(y)^2 - 2(-1)^{(m-1)/2} T_n(x) T_m(y) \sin\delta - \cos^2\delta = 0 \tag{5}$$

if m is odd and $\cos(\delta) \neq 0$, and

$$\sin\delta T_n(x) - (-1)^{(m-1)/2} T_m(y) = 0 \tag{6}$$

if m is odd and $\cos\delta = 0$.

Proof. We will prove (3) and (4), the cases when m is even. When m is odd, (5) and (6) are obtained similarly.

Applying (2) with m even as well as the characteristic property of Chebyshev polynomials we obtain

$$\begin{aligned} T_m(y) &= T_m(\sin(nt + q)) = (-1)^{m/2} \cos(mnt + mq), \\ T_n(x) &= T_n(\cos(mt + p)) = \cos(mnt + np). \end{aligned}$$

Applying the formula for the cosine of the sum of two angles we get

$$\begin{aligned} \cos mq \cos mnt - \sin mq \sin mnt &= (-1)^{m/2} T_m(y) \\ \cos np \cos mnt - \sin np \sin mnt &= T_n(x). \end{aligned} \tag{7}$$

This is a linear system in $\cos mnt$ and $\sin mnt$, with determinant

$$\begin{aligned} \Delta = \begin{vmatrix} \cos mq & -\sin mq \\ \cos np & -\sin np \end{vmatrix} &= \sin mq \cos np - \cos mq \sin np \\ &= \sin(mq - np) = \sin\delta. \end{aligned}$$

If $\sin\delta \neq 0$, applying Cramer's rule we obtain the solution

$$\begin{aligned} \cos mnt &= \frac{\sin mq}{\sin\delta} T_n(x) - \frac{(-1)^{m/2} \sin np}{\sin\delta} T_m(y) \\ \sin mnt &= \frac{\cos mq}{\sin\delta} T_n(x) - \frac{(-1)^{m/2} \cos np}{\sin\delta} T_m(y). \end{aligned}$$

Substituting in the identity $\cos^2 mnt + \sin^2 mnt = 1$ and observing that according to the formula for the cosine of the difference of two angles, the coefficient of $T_m(x)T_n(y)$

is

$$\frac{-2(-1)^{m/2}}{\sin^2\delta}(\cos mq\cos np+\sin mq\sin np)=\frac{-2(-1)^{m/2}}{\sin^2\delta}\cos(mq-np),$$

we obtain (3) after multiplying the equation by $\sin^2\delta$.

If $\sin\delta=0$, the system (7) has a solution if

$$\frac{\cos mq}{\cos np}=\frac{-\sin mq}{-\sin np}=\frac{(-1)^{m/2}T_m(y)}{T_n(x)},$$

and then

$$\begin{aligned}\cos mq T_n(x)&=(-1)^{m/2}\cos np T_m(y)\\ \sin mq T_n(x)&=(-1)^{m/2}\sin np T_m(y).\end{aligned}$$

Adding the equations after multiplying the first by $\cos np$ and the second by $\sin np$ gives

$$(\cos mq\cos np+\sin mq\sin np)T_n(x)=(-1)^{m/2}(\cos^2 np+\sin^2 np)T_m(y),$$

or

$$\cos(mq-np)T_n(x)=(-1)^{m/2}T_m(y),$$

which is (4).

For example, the non-degenerate Lissajous curve with m even and $\sin\delta=1$,

$$\begin{aligned}x&=3\cos 2t\\ y&=\sqrt{2}\sin\left(3t+\frac{\pi}{4}\right)\end{aligned}$$

has the implicit equation

$$T_3\left(\frac{x}{3}\right)^2+T_2\left(\frac{y}{\sqrt{2}}\right)^2=1$$

or, simplifying,

$$\left(\frac{4x^3}{27}-x\right)^2+(y^2-1)^2=1.$$

Some other examples are

Parametric equations	Implicit equation
$x=\cos 2t$ $y=\sin(3t-5\pi/12)$	$(4x^3-3x)^2+(2y^2-1)^2-\sqrt{3}(4x^3-3x)(2y^2-1)=\frac{1}{4}$
$x=\cos(3t+\pi/4)$ $y=\sin t$	$x^2+(4y^3-3y)^2-\sqrt{2}x(4y^3-3y)=\frac{1}{2}$

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$$
\begin{array}{ll}
x = \cos 4t & 4x^3 - 3x = 8y^4 - 8y^2 + 1 \\
y = \sin 3t & \\[1ex]
x = \cos 5t & 3x - 4x^3 = 16y^5 - 20y^3 + 5y \\
y = \sin(3t - \pi/2) &
\end{array}
$$

It is natural to ask if the converse of the theorem is true, i.e., if a point satisfying any one of the equations (3)–(6) is on a Lissajous figure. The answer to this question is "yes" for the non-degenerate cases (3) and (5) and "not always" for the degenerate cases (4) and (6). The proofs will not be given here (for (3) and (5) they essentially amount to working the proof of the theorem backwards) but are available from the author.

Equations (3)–(6) are defined for any positive integers m and n. If

$$d = \gcd(m, n) > 1,$$

each of the equations defines implicitly a curve that contains the Lissajous figure with parameters $m' = m/d$, $n' = n/d$, $p = 0$, and $q = \delta/m'$, and also other curves. With regard to these equations we put forward the following conjectures:

Conjecture I. Each of the implicit equations (3)–(6) determines a curve formed by a finite number of Lissajous figures.

Conjecture II. The polynomials in two variables that define equations (2)–(5) are irreducible in **R** if and only if m and n are prime to each other. ■

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Operations on Denominators

Rick Poss (St. Norbert's College, rick.poss@snc.edu) notes that if we have a fraction with a radical in its denominator we *rationalize* the denominator and asks if this means that if we have a fraction with a complex number in its denominator we *realize* it.